Messiah College
Calculus 1 Placement Exam Topics and Review: Key

Answers to problems in the text are listed in the back of the course textbook: *Calculus, 9th edition* by Larson, Hostetler, and Edwards. Solutions to additional problems from the review are given below.

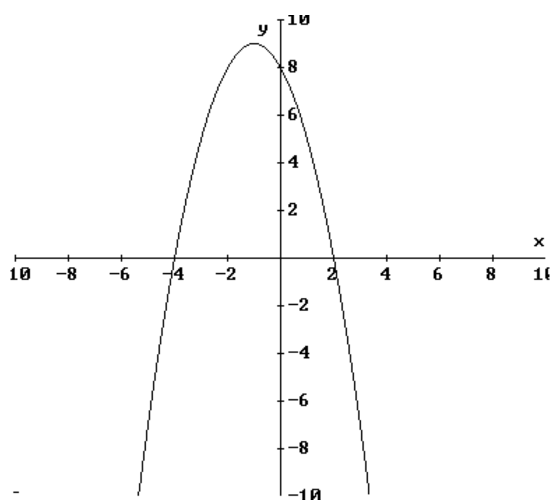
<u>Graphs and Models</u>

1. Equations and graphs of parabolas

ex. Find the vertex of the parabola $y = 2x^2 - x + 3$. Where does this parabola intersect the x-axis? y-axis?

<u>Answer:</u> The first coordinate of the vertex of a parabola is always $x = \frac{-b}{2a}$, when the equation is in standard form $y=ax^2+bx+c$. So the x-coordinate of the vertex is $x=\frac{1}{4}$. To find the y-coordinate of the vertex, plug x=0.25 into the equation of the parabola, to find y=2.875. The vertex is at (0.25,2.875). The parabola opens up, since a>0, and since the vertex is the lowest point, we know that the graph has no x-intercepts. The y-intercept is at y=3.

ex. Find the equation of the parabola shown below.



<u>Answer:</u> The parabola above intercepts the x-axis at x= -4 and x=2, so its equation must have factors (x+4) and (x-2). The equation $y=(x+4)(x-2)$ is $y=x^2+2x-8$ in standard form, which opens up (a>0), not down as required. Try $y = -(x^2+2x-8) = -x^2-2x+8$. This equation has y-

intercept at y=8 and vertex at (-1,9), which matches the graph. The correct equation is y= -x²-2x+8.

2. Quadratic equations and the quadratic formula.

ex. Solve the following quadratic equations:

i. $2x^2 + 2x - 3 = 0$.

<u>Answer:</u> Use the quadratic formula to find

$$x = \frac{-1+\sqrt{7}}{2}\ and\ x = \frac{-1-\sqrt{7}}{2}$$

ii. $3x^2 - x = 3$

<u>Answer:</u> Subtract 3 from both sides and use the quadratic formula to find $x = \frac{1+\sqrt{37}}{6}\ and\ x = \frac{1-\sqrt{37}}{6}$

3. Find solution sets for simultaneous linear equations.

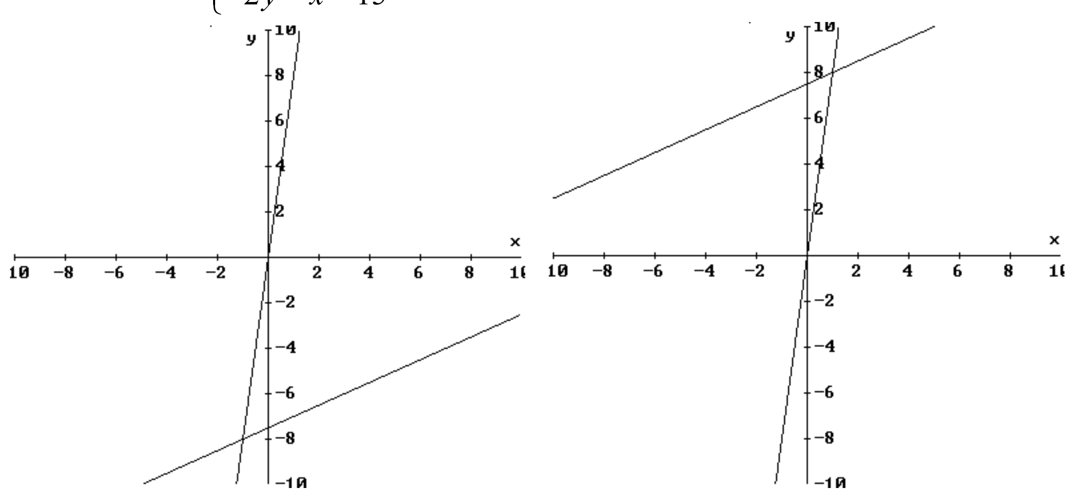
ex. Solve the following system of equations:

$$\begin{cases} y = 2x + 3 \\ y = -4x + 8 \end{cases}$$

<u>Answer:</u> Subtract the second equation from the first and solve for x to find $x = \frac{5}{6}$**. Plug this back in to either equation to get y=**$\frac{14}{3}$**.**

ex. Which of the following graphs shows the system of equations

$$\begin{cases} \frac{x+y}{3} = y - 5x \\ 2y - x = 15 \end{cases}$$ and its solution?



Answer: **The intersection point is (1,8), so the second graph is correct. (The second equation is a line with slope ½ and y-intercept at 7.5. This line is drawn correctly only on the second graph.)**

4. Find solution sets for simultaneous equations in general.

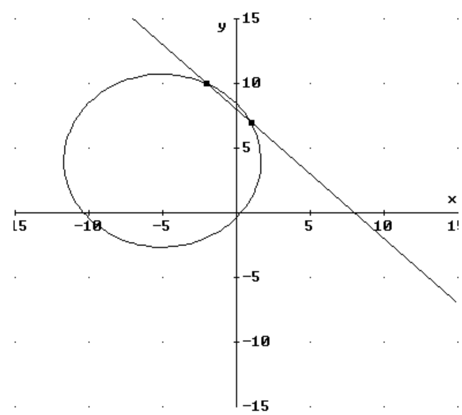
ex Solve the following systems of equations:

$$y = -5x + 6$$
$$y = x^2$$

Answer: **Substitute x^2 for y in the first equation and then factor to find two solutions, at x= -6 (y=36) and x=1 (y=1). The intersection points are (-6,36) and (1,1)**

ex. Find all intersections of the circle $x^2 + 10x + y^2 - 8y = 4$ with the line $x + y = 8$. Sketch the two curves and label the points of intersection.

Answer: **Substitute 8-x for y in the first equation, simplify, and factor, to find two solutions at x= -2 (y=10) and x=1 (y=7).**



Linear Models and Rates of Change

1. linear equations including perpendicular and parallel lines.

ex. Identify the slope and the intercepts for the equation $3x + 4y = 12$. Sketch the graph of this equation.

Answer: **The slope is $-\frac{3}{4}$.** The y-intercept is y=3 and the x-intercept is x=4.

ex. Find the equation of the line passing through the point (-1,3) and parallel to the line $3x + 4y = 12$.

Answer: **The line will have slope** $-\frac{3}{4}$ **(see above problem). Using the point-slope form, the equation is** $y-3=-\frac{3}{4}(x+1)$**, or** $y=-\frac{3}{4}x+\frac{9}{4}$**.**

ex. Find the equation of the line passing through the points (1,-1) and (1,1000).
Answer: **The line has an undefined slope and is therefore a vertical line, whose equation is x=1.**

Functions and Their Graphs

(pp. 19-29 Exercises 3,7,9,11,15,33,35)

1. Domain of a function:

ex. Find the domain of $\frac{1}{\sqrt{x^2-16}}$

Answer: **The set of all x-values for which the expression is defined is the set for which** $x^2-16>0$**. This is the set (-∞,-4) U (4, ∞).**

2. Functional notation and equations as functions:

ex. For the function $f(x)=\frac{1}{\sqrt{x^2-16}}$, find:

a) f(5) b) f(a) c) f(a+3)

Answers: **a) f(5)=**$\frac{1}{3}$ **b) f(x)=**$\frac{1}{\sqrt{a^2-16}}$

c) f(a+3)=$\frac{1}{\sqrt{(a+3)^2-16}}=\frac{1}{\sqrt{a^2+6a-7}}$

3. Use translations and shifting to describe the graphs of functions.

ex. Use the graph of $f(x)=x^2$ to sketch the following graphs:

a) $f(x+2)=(x+2)^2$ b) $f(x)+2=x^2+2$ c) $2f(x)-3=2x^2-3$

Answers:

a) The graph of $y=(x+2)^2$ **is the same as the graph of** $y=x^2$ **shifted to the left 2 units.**

b) The graph of $y=x^2+2$ **is the same as the graph of** $y=x^2$ **shifted up 2 units.**

c) The graph of $y=2x^2-3$ is the same as the graph of $y=x^2$ stretched vertically by a factor of 2 and shifted down 3 units.

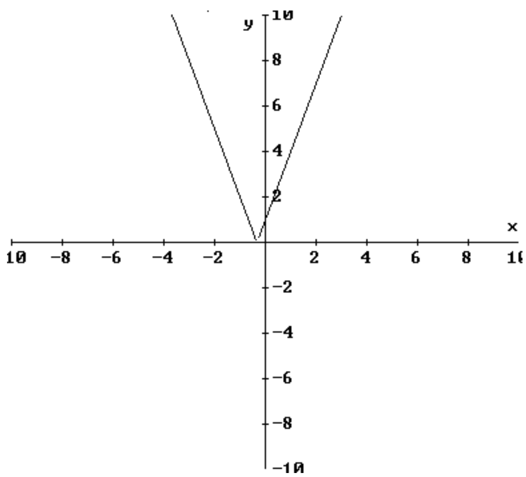
<u>Real Numbers and the Real Line</u>

1. Know the definition of absolute value and solve equalities and inequalities involving absolute value:

   ex. $|2x+5|=3$

   <u>Answer:</u> There are two solutions: x= -1 and x= -4.

   Shade the portion of the x-axis that corresponds to the solution of $|3x+1|>5$, given the graph of y=|3x+1| below.



   <u>Answer:</u> The solution set of $|3x+1|>5$ is $(-\infty,-2)$ U $(\frac{4}{3},\infty)$. Shade these two portions of the graph above, and you can see that the y-values are greater than 5 on these two regions of the x-axis.

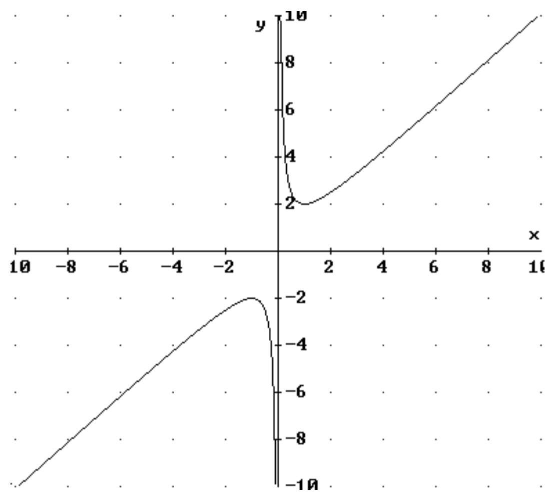
2. Solve inequalities in general:

   ex. Solve $x^2+1>4x$

   <u>Answer:</u> Subtract 4x from both sides and use the quadratic formula to find that $x^2-4x+1=0$ at x=$2-\sqrt{3}$ *and* $2+\sqrt{3}$. Place these points on a number line and test a point in each interval to find that the inequality is satisfied in the set $\left(-\infty,2-\sqrt{3}\right)U\left(2+\sqrt{3},\infty\right)$

Shade the portion of the x-axis that corresponds to the solution of $x + \frac{1}{x} \geq 1$,

given the graph of $y = x + \frac{1}{x}$ below.



Answer: The graph above clearly shows that the function $x + \frac{1}{x}$ is always above y=1 when x>0 and below y=1 when x<0 (and undefined when x=0). The x-axis should be shaded to the right of x=0, with an open circle at x=0.

ex. Solve $x^2 + x - 2 \leq 0$

Answer: The expression on the left factors: (x+2)(x-1), which is equal to 0 at x= -2 and x=1. The solution is the set [-2,1].

The Cartesian Plane

1. Distance and midpoint formulas

   ex. Find the distance between (2,3) and (-1,4).

   Answer: $\sqrt{10}$

   ex. Do the following points form a right triangle: (4,0), (2,1), (-1,-5)?

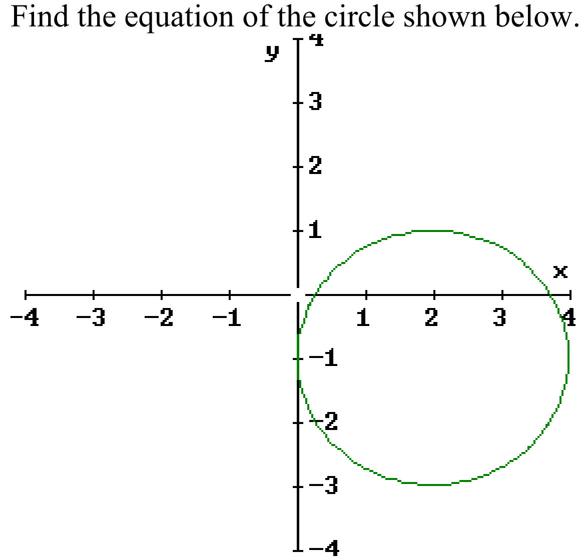
   Answer: Graphing these points and connecting them suggests that the line connecting (4,0) and (2,1) may be perpendicular to the line connecting (2,1) and (-1,-5). The first line has slope –0.5, and the second has slope 2. So they are perpendicular and yes, the points form a right triangle.

2. Equation of circles and completing the square:

   ex. Find the radius and center of the circle $x^2 - 4x + y^2 + 2y = -1$. Sketch the graph.

   Answer: Complete the square to get the equation in the standard form of a circle: $(x-2)^2+(y+1)^2=4$. The circle has radius 2 and center (2,-1).

   ex. Find the equation of the circle shown below.



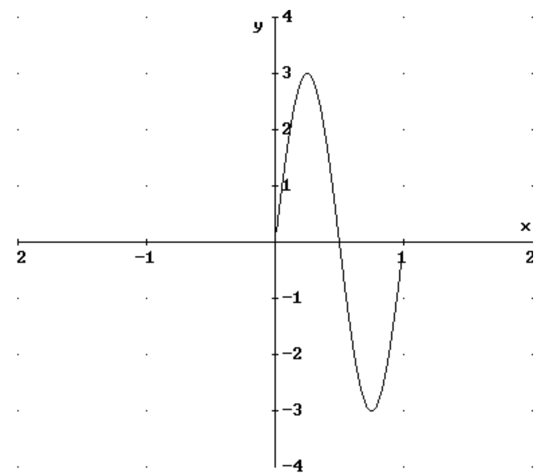
Answer: **The graph shows a center at (2,-1) and radius 2. The equation is $(x-2)^2+(y+1)^2=4$, as in the problem above.**

Review of Trigonometric Functions

1. the amplitude and period of functions of the form $y = A\sin Bx$ or $y = A\cos Bx$.

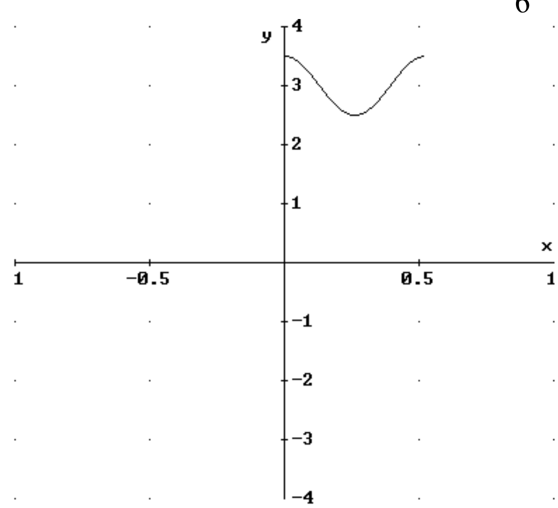
ex. Sketch the graph of $y = 3\sin 2\pi x$. What is the amplitude and period of this function.

Answer: The amplitude is 3 and the period is 1.



ex. Sketch the graph of $y = 3 + .5\cos 12x$. State the amplitude and period of this function.

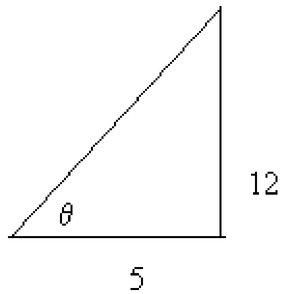
Answer: The amplitude is 0.5 and the period is $\frac{\pi}{6}$.



ex. If $\tan\theta = \sqrt{3}$ and $0 \le \theta \le \frac{\pi}{2}$, what is θ in radians?

Answer: $\theta = \frac{\pi}{3}$ radians.

ex. What are $\sin\theta, \cos\theta,$ and $\sec\theta$?



Answer: sin θ=$\frac{12}{13}$, cos **θ=**$\frac{5}{13}$**, sec θ=**$\frac{13}{5}$

2. simple trigonometric equations.

ex. Solve the following equations:

i. $\sin\theta = \cos\theta$.

Answer: θ=$\frac{\pi}{4}$**+πk for integer k**

ii. $\sin\theta\cos\theta = .5\cos\theta$.

Answer: Factor into (cos θ)(0.5-sin θ)=0. The first factor is 0 for θ=$\frac{\pi}{2}$**+kπ for integer k, and the second factor is 0 for θ=**$\frac{\pi}{6}+2\pi k$ **and θ=**$\frac{5\pi}{6}+2\pi k$**.**

Other Problems

1. Simplify the following:

ex. $\frac{(3y^3)(2y^2)^2}{(y^4)^3}$

Answer: $\frac{12}{y^5}$

ex. $\left(\frac{4a^2b}{a^3b^2}\right)\left(\frac{5a^2b}{2b^4}\right)$

Answer: $\frac{10a}{b^4}$

ex. $\left(5x^2y^{-2}\right)\left(4x^{-5}y^4\right)$

Answer: $\dfrac{20y^2}{x^3}$

ex. $\left(\dfrac{-8x^3}{y^{-6}}\right)^{2/3}$

Answer: $\mathbf{4x^2y^4}$

2. Perform the indicated operation and simplify:

ex. $(3u-1)(u-4)+7u(u+1)$

Answer: $\mathbf{10u^2-6u+4}$

ex. $\dfrac{5a^2+12a+4}{a^4-16} \div \dfrac{25a^2+20a+4}{a^2-2a}$

Answer: $\dfrac{a}{(a^2+4)(5a+2)}$

ex $\dfrac{4x}{3x-4}+\dfrac{8}{3x^2-4x}+\dfrac{2}{x}$

Answer: $\dfrac{4x+6}{3x-4}$

ex. $\dfrac{9x^2-4}{3x^2-5x+2}\cdot\dfrac{9x^4-6x^3+4x^2}{27x^4+8x}$

Answer: $\dfrac{x}{x-1}$

ex. $\dfrac{\dfrac{x}{y^2}-\dfrac{y}{x^2}}{\dfrac{1}{y^2}-\dfrac{1}{x^2}}$

Answer: $\dfrac{x^2+xy+y^2}{x+y}$

3. Factor polynomials.

ex. Factor the following:

i. x^4-16
Answer: $\mathbf{(x^2+4)(x-2)(x+2)}$

ii. $2x^4-5x^3-3x^2$
Answer: $\mathbf{x^2(2x+1)(x-3)}$

iii. $3x^3+3x^2-27x-27$
Answer: $\mathbf{3(x+1)(x+3)(x-3)}$