

Messiah College

Calculus 1 Placement Exam Topics and Review: **Key**

Solutions to problems in the text are listed in the back of the course textbook:
Calculus, 8th edition by Larson, Hostetler, and Edwards.

Graphs and Models

1. Equations and graphs of parabolas

ex. Find the vertex of the parabola $y = 2x^2 - x + 3$. Where does this parabola intersect the x-axis? y-axis?

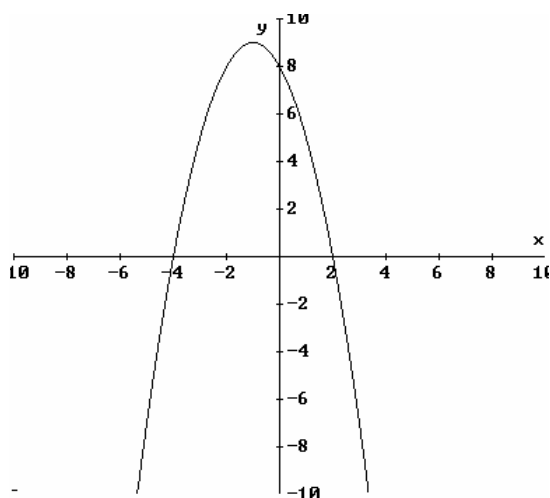
Answer: The first coordinate of the vertex of a parabola is always

$x = \frac{-b}{2a}$, when the equation is in standard form $y = ax^2 + bx + c$. So the x-

coordinate of the vertex is $x = \frac{1}{4}$. To find the y-coordinate of the

vertex, plug $x = 0.25$ into the equation of the parabola, to find $y = 2.875$. The vertex is at $(0.25, 2.875)$. The parabola opens up, since $a > 0$, and since the vertex is the lowest point, we know that the graph has no x-intercepts. The y-intercept is at $y = 3$.

ex. Find the equation of the parabola shown below.



Answer: The parabola above intercepts the x-axis at $x = -4$ and $x = 2$, so its equation must have factors $(x+4)$ and $(x-2)$. The equation $y = (x+4)(x-2)$ is $y = x^2 + 2x - 8$ in standard form, which opens up ($a > 0$), not down as required. Try $y = -(x^2 + 2x - 8) = -x^2 - 2x + 8$. This equation has y-intercept at $y = 8$ and vertex at $(-1, 9)$, which matches the graph. The correct equation is $y = -x^2 - 2x + 8$.

2. Quadratic equations and the quadratic formula.

ex. Solve the following quadratic equations:

i. $2x^2 + 2x - 3 = 0.$

Answer: Use the quadratic formula to find

$$x = \frac{-1 + \sqrt{7}}{2} \text{ and } x = \frac{-1 - \sqrt{7}}{2}$$

ii. $3x^2 - x = 3$

Answer: Subtract 3 from both sides and use the quadratic

formula to find $x = \frac{1 + \sqrt{37}}{6}$ and $x = \frac{1 - \sqrt{37}}{6}$

3. Find solution sets for simultaneous linear equations.

ex. Solve the following system of equations:

$$\begin{cases} y = 2x + 3 \\ y = -4x + 8 \end{cases}$$

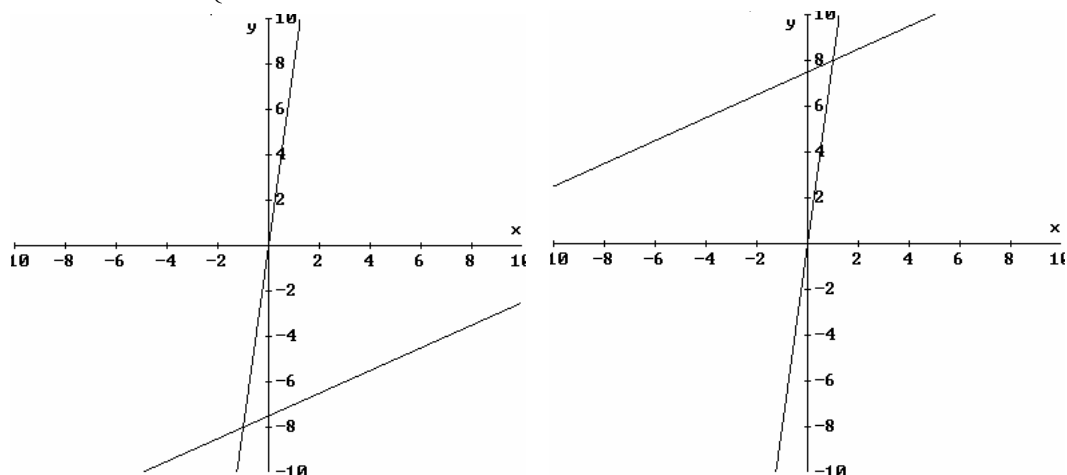
Answer: Subtract the second equation from the first and solve

for x to find $x = \frac{5}{6}$. Plug this back in to either equation to get

$$y = \frac{14}{3}.$$

ex. Which of the following graphs shows the system of equations

$$\begin{cases} \frac{x + y}{3} = y - 5x \\ 2y - x = 15 \end{cases} \text{ and its solution?}$$



Answer: The intersection point is (1,8), so the second graph is correct. (The second equation is a line with slope $\frac{1}{2}$ and y-intercept at 7.5. This line is drawn correctly only on the second graph.)

4. Find solution sets for simultaneous equations in general.

ex Solve the following systems of equations:

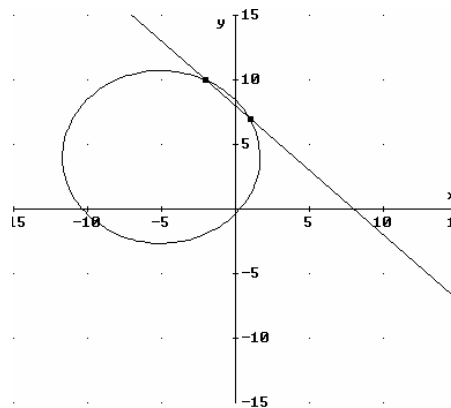
$$y = -5x + 6$$

$$y = x^2$$

Answer: Substitute x^2 for y in the first equation and then factor to find two solutions, at $x = -6$ ($y = 36$) and $x = 1$ ($y = 1$). The intersection points are $(-6, 36)$ and $(1, 1)$

ex. Find all intersections of the circle $x^2 + 10x + y^2 - 8y = 4$ with the line $x + y = 8$. Sketch the two curves and label the points of intersection.

Answer: Substitute $8 - x$ for y in the first equation, simplify, and factor, to find two solutions at $x = -2$ ($y = 10$) and $x = 1$ ($y = 7$).



Linear Models and Rates of Change

1. linear equations including perpendicular and parallel lines.

ex. Identify the slope and the intercepts for the equation $3x + 4y = 12$. Sketch the graph of this equation.

Answer: The slope is $-\frac{3}{4}$. The y-intercept is $y = 3$ and the x-intercept is $x = 4$.

ex. Find the equation of the line passing through the point $(-1, 3)$ and parallel to the line $3x + 4y = 12$.

Answer: The line will have slope $-\frac{3}{4}$ (see above problem). Using the point-slope form, the equation is $y - 3 = -\frac{3}{4}(x + 1)$, or $y = -\frac{3}{4}x + \frac{9}{4}$.

- ex. Find the equation of the line passing through the points (1,-1) and (1,1000).

Answer: The line has an undefined slope and is therefore a vertical line, whose equation is $x=1$.

Functions and Their Graphs

(pp. 19-29 Exercises 3,7,9,11,15,33,35)

1. Domain of a function:

ex. Find the domain of $\frac{1}{\sqrt{x^2 - 16}}$

Answer: The set of all x-values for which the expression is defined is the set for which $x^2-16>0$. This is the set $(-4,-4) \cup (4, 4)$.

2. Functional notation and equations as functions:

ex. For the function $f(x) = \frac{1}{\sqrt{x^2 - 16}}$, find:

a) $f(5)$ b) $f(a)$ c) $f(a+3)$

Answers: a) $f(5) = \frac{1}{3}$ b) $f(x) = \frac{1}{\sqrt{a^2 - 16}}$

c) $f(a+3) = \frac{1}{\sqrt{(a+3)^2 - 16}} = \frac{1}{\sqrt{a^2 + 6a - 7}}$

3. Use translations and shifting to describe the graphs of functions.

ex. Use the graph of $f(x) = x^2$ to sketch the following graphs:

a) $f(x+2) = (x+2)^2$ b) $f(x)+2 = x^2+2$ c) $2f(x)-3 = 2x^2-3$

Answers:

a) The graph of $y=(x+2)^2$ is the same as the graph of $y=x^2$ shifted to the left 2 units.

b) The graph of $y=x^2+2$ is the same as the graph of $y=x^2$ shifted up 2 units.

c) The graph of $y=2x^2-3$ is the same as the graph of $y=x^2$ stretched vertically by a factor of 2 and shifted down 3 units.

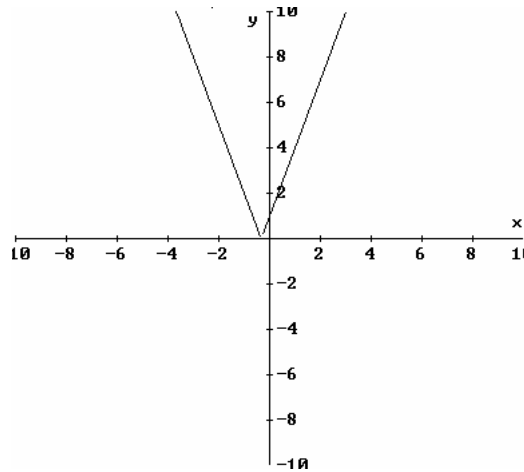
Real Numbers and the Real Line

1. Know the definition of absolute value and solve equalities and inequalities involving absolute value:

ex. $|2x + 5| = 3$

Answer: There are two solutions: $x = -1$ and $x = -4$.

Shade the portion of the x-axis that corresponds to the solution of $|3x + 1| > 5$, given the graph of $y = |3x + 1|$ below.



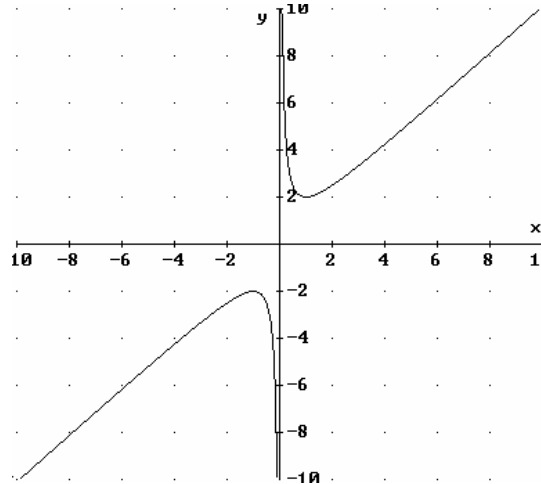
Answer: The solution set of $|3x + 1| > 5$ is $(-4, -2) \cup (\frac{4}{3}, 4)$. Shade these two portions of the graph above, and you can see that the y-values are greater than 5 on these two regions of the x-axis.

2. Solve inequalities in general:

ex. Solve $x^2 + 1 > 4x$

Answer: Subtract $4x$ from both sides and use the quadratic formula to find that $x^2 - 4x + 1 = 0$ at $x = 2 - \sqrt{3}$ and $2 + \sqrt{3}$. Place these points on a number line and test a point in each interval to find that the inequality is satisfied in the set $(-\infty, 2 - \sqrt{3}) \cup (2 + \sqrt{3}, \infty)$

Shade the portion of the x-axis that corresponds to the solution of $x + \frac{1}{x} \geq 1$, given the graph of $y = x + \frac{1}{x}$ below.



Answer: The graph above clearly shows that the function $x + \frac{1}{x}$ is always above $y=1$ when $x>0$ and below $y=1$ when $x<0$ (and undefined when $x=0$). The x-axis should be shaded to the right of $x=0$, with an open circle at $x=0$.

ex. Solve $x^2 + x - 2 \leq 0$

Answer: The expression on the left factors: $(x+2)(x-1)$, which is equal to 0 at $x = -2$ and $x = 1$. The solution is the set $[-2, 1]$.

The Cartesian Plane

1. Distance and midpoint formulas

ex. Find the distance between (2,3) and (-1,4).

Answer: $\sqrt{10}$

ex. Do the following points form a right triangle: (4,0), (2,1), (-1,-5)?

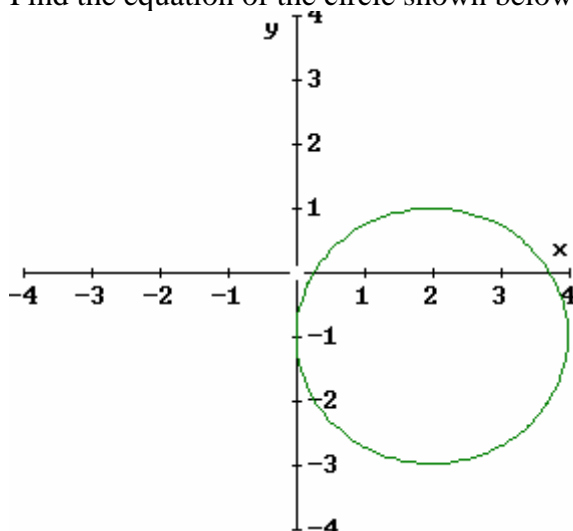
Answer: Graphing these points and connecting them suggests that the line connecting (4,0) and (2,1) may be perpendicular to the line connecting (2,1) and (-1,-5). The first line has slope -0.5 , and the second has slope 2 . So they are perpendicular and yes, the points form a right triangle.

2. Equation of circles and completing the square:

ex. Find the radius and center of the circle $x^2 - 4x + y^2 + 2y = -1$. Sketch the graph.

Answer: Complete the square to get the equation in the standard form of a circle: $(x-2)^2 + (y+1)^2 = 4$. The circle has radius 2 and center $(2,-1)$.

ex. Find the equation of the circle shown below.



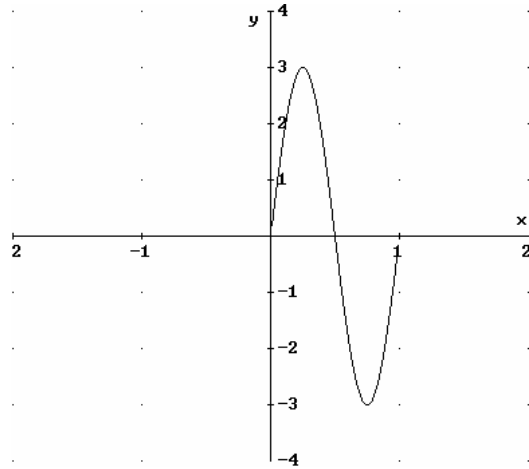
Answer: The graph shows a center at $(2,-1)$ and radius 2 .
The equation is $(x-2)^2 + (y+1)^2 = 4$, as in the problem above.

Review of Trigonometric Functions

1. the amplitude and period of functions of the form $y = A \sin Bx$ or $y = A \cos Bx$.

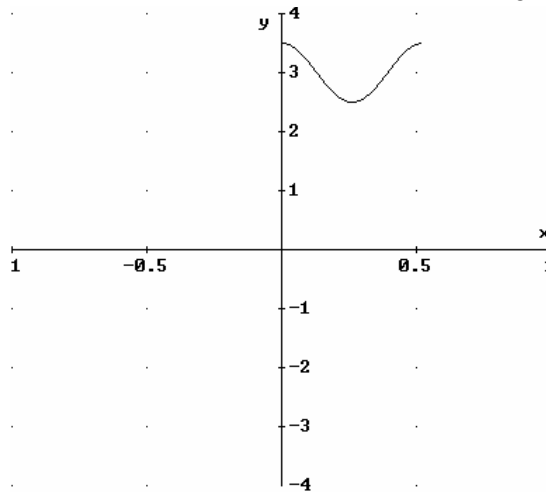
ex. Sketch the graph of $y = 3 \sin 2\pi x$. What is the amplitude and period of this function.

Answer: The amplitude is 3 and the period is 1.



ex. Sketch the graph of $y = 3 + 0.5 \cos 12x$. State the amplitude and period of this function.

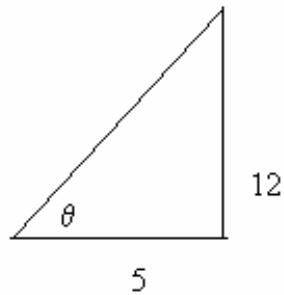
Answer: The amplitude is 0.5 and the period is $\frac{\pi}{6}$.



ex. If $\tan \theta = \sqrt{3}$ and $0 \leq \theta \leq \frac{\pi}{2}$, what is θ in radians?

Answer: $2 = \frac{\pi}{3}$ radians.

ex. What are $\sin \theta$, $\cos \theta$, and $\sec \theta$?



Answer: $\sin \theta = \frac{12}{13}$, $\cos \theta = \frac{5}{13}$, $\sec \theta = \frac{13}{5}$

2. simple trigonometric equations.

ex. Solve the following equations:

i. $\sin \theta = \cos \theta$.

Answer: $\theta = \frac{\pi}{4} + k\pi$ for integer k

ii. $\sin \theta \cos \theta = 5 \cos \theta$.

Answer: Factor into $(\cos \theta)(0.5 - \sin \theta) = 0$. The first factor is 0 for

$\theta = \frac{\pi}{2} + k\pi$ for integer k , and the second factor is 0 for $\theta = \frac{\pi}{6} + 2\pi k$

and $\theta = \frac{5\pi}{6} + 2\pi k$.

Other Problems

1. Simplify the following:

ex. $\frac{(3y^3)(2y^2)^2}{(y^4)^3}$

Answer: $\frac{12}{y^5}$

ex. $\left(\frac{4a^2b}{a^3b^2}\right)\left(\frac{5a^2b}{2b^4}\right)$

Answer: $\frac{10a}{b^4}$

ex. $(5x^2y^{-2})(4x^{-5}y^4)$

Answer: $\frac{20y^2}{x^3}$

ex. $\left(\frac{-8x^3}{y^{-6}}\right)^{2/3}$

Answer: $4x^2y^4$

2. Perform the indicated operation and simplify:

ex. $(3u-1)(u-4)+7u(u+1)$

Answer: $10u^2-6u+4$

ex. $\frac{5a^2+12a+4}{a^4-16} \div \frac{25a^2+20a+4}{a^2-2a}$

Answer: $\frac{a}{(a^2+4)(5a+2)}$

ex. $\frac{4x}{3x-4} + \frac{8}{3x^2-4x} + \frac{2}{x}$

Answer: $\frac{4x+6}{3x-4}$

ex. $\frac{9x^2-4}{3x^2-5x+2} \cdot \frac{9x^4-6x^3+4x^2}{27x^4+8x}$

Answer: $\frac{x}{x-1}$

ex. $\frac{\frac{x}{y^2} - \frac{y}{x^2}}{\frac{1}{y^2} - \frac{1}{x^2}}$

Answer: $\frac{x^2+xy+y^2}{x+y}$

3. Factor polynomials.

ex. Factor the following:

i. x^4-16
Answer: $(x^2+4)(x-2)(x+2)$

ii. $2x^4-5x^3-3x^2$
Answer: $x^2(2x+1)(x-3)$

iii. $3x^3+3x^2-27x-27$
Answer: $3(x+1)(x+3)(x-3)$